Fractional Variational Calculus with Classical and Combined Caputo Derivatives*

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Abstract

We give a proper fractional extension of the classical calculus of variations by considering variational functionals with a Lagrangian depending on a combined Caputo fractional derivative and the classical derivative. Euler–Lagrange equations to the basic and isoperimetric problems are proved, as well as transversality conditions.

Keywords: fractional derivatives; fractional variational analysis; isoperimetric problems; natural boundary conditions; Euler–Lagrange equations.

2010 Mathematics Subject Classification: 49K05; 49K21; 26A33; 34A08.

1 Introduction

Fractional calculus (FC) is a generalization of (integer) differential calculus, in the sense it deals with derivatives of real or complex order. FC was born on 30th September 1695. On that day, L'Hôpital wrote a letter to Leibniz, where

*Part of the first author's Ph.D., which is carried out at the University of Aveiro under the Doctoral Programme *Mathematics and Applications* of Universities of Aveiro and Minho. Submitted 30-Nov-2010; accepted 14-Jan-2011; for publication in *Nonlinear Analysis Series A: Theory, Methods & Applications.*

he asked about Leibniz's notation of nth order derivatives of a linear function. L'Hôpital wanted to know the result for the derivative of order $n = 1/2$. Leibniz replied that "*one day, useful consequences will be drawn*" and, in fact, his vision became a reality. The study of non-integer order derivatives rapidly became a very attractive subject to mathematicians, and many different forms of fractional (i.e., non-integer) derivative operators were introduced: the Grunwald–Letnikow, Riemann–Liouville, Caputo [15, 17, 26], and the more recent notions of Cresson [10], Jumarie [16], or Klimek [18].

The calculus of variations with fractional derivatives was born in 1996 with the work of Riewe, to better describe nonconservative systems in mechanics [27, 28]. It is a subject of strong current research due to its many applications in science and engineering, including mechanics, chemistry, biology, economics, and control theory (see, e.g., the recent papers [2, 5, 6, 7, 8, 13, 19, 21, 23, 24]).[1]

Following [25], we consider here that the highest derivative in the Lagrangian is of integer order. The main advantage of our formulation, with respect to the "pure" fractional approach adopted in the literature, is that the classical results of variational calculus can now be obtained as particular cases. We recall that the only possibility of obtaining the classical derivative y' from a fractional derivative $D^\alpha y$, $\alpha \in (0,1)$, is to take the limit when α tends to one. However, in general such a limit does not exist [29]. Differently from [25], where the fractional problems are considered in the sense of Riemann–Liouville, we consider here combined Caputo derivatives ${}^C D_\gamma^{\alpha,\beta}$. The operator ${}^C D_\gamma^{\alpha,\beta}$ extends the Caputo fractional derivatives, and was introduced for the first time in [22] as a useful tool in the description of some nonconservative models and more general classes of variational problems. More precisely, we investigate here problems of the calculus of variations with integrands depending on the independent variable x, an unknown vector-function $\mathbf{y}$, its integer order derivative $\mathbf{y}'$, and a fractional derivative ${}^C D_\gamma^{\alpha,\beta}\mathbf{y}$ given as a convex combination of the left Caputo fractional derivative of order α and the right Caputo fractional derivative of order β.

The paper is organized as follows. Section 2 presents the basic definitions and facts needed in the sequel. Our results are then stated and proved in Section 3. We discuss the fundamental concepts of a variational calculus such as the Euler–Lagrange equations for the basic (Theorem 13) and isoperimetric (Theorem 18) problems, as well as transversality conditions (Theorem 15). We end with an illustrative example of the results of the paper (Section 4).

2 Preliminaries on Fractional Calculus

In this section we present some basic necessary facts on fractional calculus. For more on the subject and applications, we refer the reader to the books [15, 17, 26].

[1]The literature on *fractional variational calculus* is vast, and we do not try to provide here a comprehensive review on the subject. We give only some representative references from 2010 and 2011. Other references can be found therein.

Definition 1 (Riemann–Liouville fractional integrals)**.** *Let $f \in L_1([a,b])$ and $0 < \alpha < 1$. The left Riemann–Liouville Fractional Integral (RLFI) of order α of a function f is defined by*

$$ {}_aI_x^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, $$

and the right RLFI by

$$ {}_xI_b^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, $$

for all $x \in [a,b]$.

Definition 2 (Left and right Riemann–Liouville fractional derivatives)**.** *The left Riemann–Liouville Fractional Derivative (RLFD) of order α of a function f, denoted by ${}_aD_x^\alpha f$, is defined by*

$$ {}_aD_x^\alpha f(x) := \frac{d}{dx}{}_aI_x^{1-\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f(t)dt, $$

$x \in [a,b]$. Similarly, the right RLFD of order α of a function f, denoted by ${}_xD_b^\alpha f$, is defined by

$$ {}_xD_b^\alpha f(x) := -\frac{d}{dx}{}_xI_b^{1-\alpha} f(x) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (t-x)^{-\alpha} f(t)dt, $$

$x \in [a,b]$.

Definition 3 (Caputo fractional derivatives)**.** *Let $f \in AC([a,b])$, where $AC([a,b])$ represents the space of absolutely continuous functions on the interval $[a,b]$. The left Caputo Fractional Derivative (CFD) is defined by*

$$ {}_a^CD_x^\alpha f(t) := {}_aI_x^{1-\alpha} \left(\frac{d}{dt} f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} \frac{d}{dt} f(t)dt, $$

$x \in [a,b]$, and the right CFD by

$$ {}_xD_b^\alpha f(x) := {}_xI_b^{1-\alpha} \left(-\frac{d}{dt} f\right)(x) = \frac{-1}{\Gamma(1-\alpha)} \int_x^b (t-x)^{-\alpha} \frac{d}{dt} f(t)dt, $$

$x \in [a,b]$, where α is the order of the derivative.

Theorem 4 (Fractional integration by parts [17])**.** *Let $p \geq 1$, $q \geq 1$, and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$. If $g \in L_p([a,b])$ and $f \in L_q([a,b])$, then the following formula for integration by parts holds:*

$$ \int_a^b g(x){}_aI_x^\alpha f(x)dx = \int_a^b f(x){}_xI_b^\alpha g(x)dx. $$

Definition 5 (The combined fractional derivative ${}^C D_\gamma^{\alpha,\beta}$ [22]). *Let $\alpha, \beta \in (0,1)$ and $\gamma \in [0,1]$. The combined fractional derivative operator ${}^C D_\gamma^{\alpha,\beta}$ is given by*

$$ {}^C D_\gamma^{\alpha,\beta} := \gamma {}_a^C D_x^\alpha + (1-\gamma) {}_x^C D_b^\beta . $$

Remark 6. *The combined fractional derivative coincides with the right CFD in the case $\gamma = 0$, i.e., ${}^C D_0^{\alpha,\beta} f(x) = {}_x^C D_b^\alpha f(x)$. For $\gamma = 1$ one gets the left CFD: ${}^C D_1^{\alpha,\beta} f(x) = {}_a^C D_x^\alpha f(x)$.*

For $\mathbf{f} = [f_1, \ldots, f_N] : [a,b] \to \mathbb{R}^N$, $N \in \mathbb{N}$, and $f_i \in AC([a,b])$, $i = 1, \ldots, N$, we put

$$ {}^C D_\gamma^{\alpha,\beta} \mathbf{f}(x) := \left[{}^C D_\gamma^{\alpha,\beta} f_1(x), \ldots, {}^C D_\gamma^{\alpha,\beta} f_N(x) \right]. $$

In the discussion to follow, we also need the following formula for fractional integrations by parts [22]:

$$ \int_a^b g(x) {}^C D_\gamma^{\alpha,\beta} f(x) dx = \int_a^b f(x) D_{1-\gamma}^{\beta,\alpha} g(x) dx + \left[\gamma f(x) {}_x I_b^{1-\alpha} g(x) - (1-\gamma) f(x) {}_a I_x^{1-\beta} g(x) \right]_{x=a}^{x=b}, \quad (1) $$

where $D_{1-\gamma}^{\beta,\alpha} := (1-\gamma) {}_a D_x^\beta + \gamma {}_x D_b^\alpha$.

3 Main Results

Consider the following functional:

$$ \mathcal{J}(\mathbf{y}) = \int_a^b L\left(x, \mathbf{y}(x), \mathbf{y}'(x), {}^C D_\gamma^{\alpha,\beta} \mathbf{y}(x)\right) dx, \quad (2) $$

where $x \in [a,b]$ is the independent variable; $\mathbf{y}(x) \in \mathbb{R}^N$ is a real vector variable; $\mathbf{y}'(x) \in \mathbb{R}^N$ with $\mathbf{y}'$ the first derivative of $\mathbf{y}$; ${}^C D_\gamma^{\alpha,\beta} \mathbf{y}(x) \in \mathbb{R}^N$ stands for the combined fractional derivative of $\mathbf{y}$ evaluated in x; and $L \in C^1\left([a,b] \times \mathbb{R}^{3N}; \mathbb{R}\right)$. Let $\mathbf{D}$ denote the set of all functions $\mathbf{y} : [a,b] \to \mathbb{R}^N$ such that $\mathbf{y}'$ and ${}^C D_\gamma^{\alpha,\beta} \mathbf{y}$ exist and are continuous on the interval $[a,b]$. We endow $\mathbf{D}$ with the norm

$$ \|\mathbf{y}\|_{1,\infty} := \max_{a \le x \le b} \|\mathbf{y}(x)\| + \max_{a \le x \le b} \|\mathbf{y}'(x)\| + \max_{a \le x \le b} \left\| {}^C D_\gamma^{\alpha,\beta} \mathbf{y}(x) \right\|, $$

where $\|\cdot\|$ is a norm in $\mathbb{R}^N$. Along the work, we denote by $\partial_i K$, $i = 1, \ldots, M$ ($M \in \mathbb{N}$), the partial derivative of a function $K : \mathbb{R}^M \to \mathbb{R}$ with respect to its ith argument. Let $\lambda \in \mathbb{R}^r$. For simplicity of notation we introduce the operators $[\cdot]_\gamma^{\alpha,\beta}$ and ${}_\lambda \{\cdot\}_\gamma^{\alpha,\beta}$ defined by

$$ [\mathbf{y}]_\gamma^{\alpha,\beta}(x) := \left(x, \mathbf{y}(x), \mathbf{y}'(x), {}^C D_\gamma^{\alpha,\beta} \mathbf{y}(x)\right), $$
$$ {}_\lambda \{\mathbf{y}\}_\gamma^{\alpha,\beta}(x) := \left(x, \mathbf{y}(x), \mathbf{y}'(x), {}^C D_\gamma^{\alpha,\beta} \mathbf{y}(x), \lambda_1, \ldots, \lambda_r\right). $$

3.1 The Euler–Lagrange equation

We begin with the following problem of the fractional calculus of variations.

Problem 1. *Find a function $y \in \mathbf{D}$ for which the functional* (2)*, i.e.,*

$$\mathcal{J}(y) = \int_a^b L[y]_\gamma^{\alpha,\beta}(x)dx, \tag{3}$$

subject to given boundary conditions

$$y(a) = \boldsymbol{y}^a, \quad y(b) = \boldsymbol{y}^b, \tag{4}$$

$\boldsymbol{y}^a, \boldsymbol{y}^b \in \mathbb{R}^N$, achieves a minimum.

Definition 7 (Admissible function)**.** *A function $y \in \mathbf{D}$ that satisfies all the constraints of a problem is said to be admissible to that problem. The set of admissible functions is denoted by $\mathcal{D}$.*

Remark 8. *For Problem 1 the constraints mentioned in Definition 7 are the boundary conditions* (4)*.*

We now define what is meant by minimum of $\mathcal{J}$ on $\mathcal{D}$.

Definition 9 (Local minimizer)**.** *A function $\overline{y} \in \mathcal{D}$ is said to be a local minimizer to $\mathcal{J}$ on $\mathcal{D}$ if there exists some $\delta > 0$ such that*

$$\mathcal{J}(\overline{y}) - \mathcal{J}(y) \leq 0$$

for all functions $y \in \mathcal{D}$ with $\|y - \overline{y}\|_{1,\infty} < \delta$.

Similarly to the classical calculus of variations, a necessary optimality condition to Problem 1 is based on the concept of variation.

Definition 10 (First variation)**.** *The first variation of $\mathcal{J}$ at $y \in \mathbf{D}$ in the direction $\mathbf{h} \in \mathbf{D}$ is defined by*

$$\delta\mathcal{J}(y; \mathbf{h}) := \lim_{\epsilon \to 0} \frac{\mathcal{J}(y + \epsilon\mathbf{h}) - \mathcal{J}(y)}{\epsilon} = \frac{\partial}{\partial\epsilon} J(y + \epsilon\mathbf{h})\Big|_{\epsilon=0},$$

provided the limit exists.

Definition 11 (Admissible variation)**.** *An admissible variation at $y \in \mathcal{D}$ for $\mathcal{J}$ is a direction $\mathbf{h} \in \mathbf{D}$, $\mathbf{h} \neq 0$, such that*

- *$\delta\mathcal{J}(y; \mathbf{h})$ exists; and*
- *$y + \epsilon\mathbf{h} \in \mathcal{D}$ for all sufficiently small ϵ.*

Theorem 12 (see, e.g., [30])**.** *Let $\mathcal{J}$ be a functional defined on $\mathcal{D}$. Suppose that y is a local minimizer to $\mathcal{J}$ on $\mathcal{D}$. Then $\delta\mathcal{J}(y; \mathbf{h}) = 0$ for each admissible variation $\mathbf{h}$ at y.*

We now state and prove the Euler–Lagrange equations for Problem 1.

Theorem 13. *If $y = (y_1, \ldots, y_N)$ is a local minimizer to Problem 1, then y satisfies the system of N Euler–Lagrange equations*

$$\partial_i L \left[y\right]_{\gamma}^{\alpha,\beta}(x) - \frac{d}{dx}\partial_{N+i} L \left[y\right]_{\gamma}^{\alpha,\beta}(x) + D_{1-\gamma}^{\beta,\alpha}\partial_{2N+i} L \left[y\right]_{\gamma}^{\alpha,\beta}(x) = 0, \tag{5}$$

$i = 2, \ldots, N+1$, *for all* $x \in [a, b]$.

Proof. Suppose that y is a solution to Problem 1 and let $\mathbf{h}$ be an arbitrary admissible variation for this problem, i.e.,

$$h_i(a) = h_i(b) = 0, \quad i = 1, \ldots, N.$$

According with Theorem 12, a necessary condition for $\mathbf{y}$ to be a minimizer is given by

$$\frac{\partial}{\partial \epsilon}\mathcal{J}\left(y + \epsilon \mathbf{h}\right)\Big|_{\epsilon=0} = 0,$$

that is,

$$\begin{aligned}\int_a^b \Bigg[&\sum_{i=2}^{N+1} \partial_i L \left[y\right]_{\gamma}^{\alpha,\beta}(x) h_{i-1}(x) + \sum_{i=2}^{N+1} \partial_{N+i} L \left[y\right]_{\gamma}^{\alpha,\beta}(x) \frac{d}{dx} h_{i-1}(x) \\ &+ \sum_{i=2}^{N+1} \partial_{2N+i} L \left[y\right]_{\gamma}^{\alpha,\beta}(x) \left({}^{C}D_{\gamma}^{\alpha,\beta} h_{i-1}(x)\right)\Bigg] dx = 0.\end{aligned} \tag{6}$$

Using the integration by parts formulas, for the classical and ${}^{C}D_{\gamma}^{\alpha,\beta}$ derivatives, in the second and third term of the integrand, we obtain

$$\begin{aligned}&\int_a^b \sum_{i=2}^{N+1} \left[\partial_i L \left[y\right]_{\gamma}^{\alpha,\beta}(x) - \frac{d}{dx}\partial_{N+i} L \left[y\right]_{\gamma}^{\alpha,\beta}(x) +^{C} D_{1-\gamma}^{\beta,\alpha}\partial_{2N+i} L \left[y\right]_{\gamma}^{\alpha,\beta}(x)\right] h_{i-1}(x) dx \\ &+ \left[\sum_{i=2}^{N+1} h_{i-1}(x)\partial_{N+i} L \left[y\right]_{\gamma}^{\alpha,\beta}(x)\right]_{x=a}^{x=b} + \gamma \left[\sum_{i=2}^{N+1} h_{i-1}(x) {}_{x}I_b^{1-\alpha}\partial_{2N+i} L \left[y\right]_{\gamma}^{\alpha,\beta}(x)\right]_{x=a}^{x=b} \\ &- (1-\gamma)\left[\sum_{i=2}^{N+1} h_{i-1}(x) {}_{a}I_x^{1-\beta}\partial_{2N+i} L \left[y\right]_{\gamma}^{\alpha,\beta}(x)\right]_{x=a}^{x=b} = 0.\end{aligned} \tag{7}$$

Since $h_i(a) = h_i(b) = 0$, $i = 1, \ldots, N$, we get

$$\int_a^b \sum_{i=2}^{N+1} \left[\partial_i L \left[y\right]_{\gamma}^{\alpha,\beta}(x) - \frac{d}{dx}\partial_{N+i} L \left[y\right]_{\gamma}^{\alpha,\beta}(x) +^{C} D_{1-\gamma}^{\beta,\alpha}\partial_{2N+i} L \left[y\right]_{\gamma}^{\alpha,\beta}(x)\right] h_{i-1}(x) dx = 0.$$

Equalities (5) follow from the application of the fundamental lemma of the calculus of variations (see, e.g., [30]). □

When the Lagrangian L does not depend on fractional derivatives, then Theorem 13 reduces to the classical result (see, e.g., [30]). The fractional Euler–Lagrange equations via Caputo derivatives that one can find in the literature, are also obtained as corollaries of Theorem 13. The next result is obtained choosing a Lagrangian that does not depend on the classical derivatives.

Corollary 14 (Theorem 6 of [22])**.** *Let $\mathbf{y} = (y_1, \ldots, y_N)$ be a local minimizer to problem*

$$\mathcal{J}(\mathbf{y}) = \int_a^b L\left(x, \mathbf{y}(x), {}^C D_\gamma^{\alpha,\beta} \mathbf{y}(x)\right) dx \longrightarrow \min$$
$$\mathbf{y}(a) = \mathbf{y}^a, \quad \mathbf{y}(b) = \mathbf{y}^b,$$

$\mathbf{y}^a$, $\mathbf{y}^b \in \mathbb{R}^N$*. Then,* $\mathbf{y}$ *satisfies the system of* N *fractional Euler–Lagrange equations*

$$\partial_i L[\mathbf{y}](x) + D_{1-\gamma}^{\beta,\alpha} \partial_{N+i} L[\mathbf{y}](x) = 0, \tag{8}$$

$i = 2, \ldots N+1$*, for all* $x \in [a, b]$.

If one considers $\gamma = 1$ (cf. Remark 6) and $N = 1$ in Corollary 14, then (8) reduces to the well known Caputo fractional Euler–Lagrange equation: if y is a local minimizer to problem

$$\mathcal{J}(y) = \int_a^b L\left(x, y(x), {}_a^C D_x^\alpha y(x)\right) dx \longrightarrow \min$$
$$y(a) = y_a, \quad y(b) = y_b,$$

then y satisfies the fractional Euler–Lagrange equation

$$\partial_2 L\left(x, y(x), {}_a^C D_x^\alpha y(x)\right) + {}_x D_b^\alpha \partial_3 L\left(x, y(x), {}_a^C D_x^\alpha y(x)\right) = 0 \tag{9}$$

for all $x \in [a, b]$ (see, e.g., [12]).

3.2 Transversality conditions

Let $l \in \{1, \ldots, N\}$. Assume now that in Problem 1 the boundary conditions (4) are substituted by

$$\mathbf{y}(a) = \mathbf{y}^a, \quad y_i(b) = y_i^b, \ i = 1, \ldots, N \text{ for } i \neq l, \text{ and } y_l(b) \text{ is free} \tag{10}$$

or

$$\mathbf{y}(a) = \mathbf{y}^a, \quad y_i(b) = y_i^b, \ i = 1, \ldots, N \text{ for } i \neq l, \text{ and } y_l(b) \leq y_l^b. \tag{11}$$

Theorem 15. *If $y = (y_1, \ldots, y_N)$ is a solution to Problem 1 with either* (10) *or* (11) *as boundary conditions instead of* (4)*, then y satisfies the system of Euler–Lagrange equations* (5)*. Moreover, under the boundary conditions* (10) *the extra transversality condition*

$$\Big[\partial_{N+l+1} L\left[y\right]_\gamma^{\alpha,\beta}(x) + \gamma {}_x I_b^{1-\alpha} \partial_{2N+l+1} L\left[y\right]_\gamma^{\alpha,\beta}(x) - (1-\gamma) {}_a I_x^{1-\beta} \partial_{2N+l+1} L\left[y\right]_\gamma^{\alpha,\beta}(x)\Big]_{x=b} = 0 \tag{12}$$

holds; under the boundary conditions (11) *the extra transversality condition*

$$\Big[\partial_{N+l+1}L\,[y]^{\alpha,\beta}_{\gamma}(x)+\gamma_x I_b^{1-\alpha}\partial_{2N+l+1}L\,[y]^{\alpha,\beta}_{\gamma}(x) \\ -(1-\gamma)_a I_x^{1-\beta}\partial_{2N+l+1}L\,[y]^{\alpha,\beta}_{\gamma}(x)\Big]_{x=b}\leq 0 \quad (13)$$

holds, with (12) *taking place if* $y_l(b)<y_l^b$.

Proof. The fact that the system of Euler–Lagrange equations (5) is satisfied is a simple consequence of the proof of Theorem 13 (one can always restrict to the subclass of functions $\mathbf{h}\in\mathbf{D}$ for which $h_i(a)=h_i(b)=0$, $i=1,\ldots,N$). Let us assume that the boundary conditions are (10). Condition (12) follows from (6). Suppose now that the boundary conditions are (11). Then, there are two cases to consider. (i) If $y_l(b)<y_l^b$, then there are admissible neighboring paths with terminal value both above and below $y_l(b)$, so that $h_l(b)$ can take either sign. Therefore, the transversality condition is (12). (ii) Let $y_l(b)=y_l^b$. In this case neighboring paths with terminal value $\tilde{y}_l\leq y_l(b)$ are considered. Choose h_l such that $h_l(b)\geq 0$. Then, $\epsilon\leq 0$ and the transversality condition, which has its root in the first order condition (7), must be changed to an inequality. We obtain (13). □

When the Lagrangian does not depend on fractional derivatives, then the left hand side of (12) and (13) reduce to the classical expression $\partial_{N+l+1}L\left(x,\mathbf{y}(x),\mathbf{y}'(x)\right)$ (for instance, when $N=1$ and $y(a)$ is fixed with $y(b)$ free, then we get the well known boundary condition $\partial_3 L\left(b,y(b),y'(b)\right)=0$). In the particular case when the Lagrangian does not depend on the classical derivatives, $\gamma=1$, $N=1$, and we have boundary conditions (10), then one obtains from Theorem 15 the following result of [1].

Corollary 16 (cf. Theorem 1 of [1])**.** *If y is a local minimizer to problem*

$$\mathcal{J}(y)=\int_a^b L\left(x,y(x),{}_a^C D_x^{\alpha}y(x)\right)dx\longrightarrow\min$$
$$y(a)=y_a\quad (y(b)\ \textit{is free}),$$

then y satisfies the fractional Euler–Lagrange equation (9)*. Moreover,*

$$\left[{}_x I_b^{1-\alpha}\partial_3 L(x,y(x),{}_a^C D_x^{\alpha}y(x))\right]_{x=b}=0.$$

3.3 The isoperimetric problem

We now consider the following problem of the calculus of variations.

Problem 2. *Minimize functional* (3) *subject to given boundary conditions* (4) *and r isoperimetric constraints*

$$\mathcal{G}^j(y)=\int_a^b G^j\,[y]^{\alpha,\beta}_{\gamma}(x)dx=\xi_j,\quad j=1,\ldots,r, \tag{14}$$

where $G^j \in C^1\left([a,b] \times \mathbb{R}^{3N}; \mathbb{R}\right)$ *and* $\xi_j \in \mathbb{R}$ *for* $j = 1, \ldots, r$*.*

Problems of the type of Problem 2, where some integrals are to be given a fixed value while another one is to be made a maximum or a minimum, are called *isoperimetric problems*. Such variational problems have found a broad class of important applications throughout the centuries, with numerous useful implications in astronomy, geometry, algebra, analysis, and engineering. For references and recent advancements on the subject, we refer the reader to [3, 4, 11, 20]. Here, in order to obtain necessary optimality conditions for the combined fractional isoperimetric problem (Problem 2), we make use of the following theorem.

Theorem 17 (see, e.g., Theorem 2 of [14] on p. 91)**.** *Let* $\mathcal{J}, \mathcal{G}^1, \ldots, \mathcal{G}^r$ *be functionals defined in a neighborhood of* y *and having continuous first variations in this neighborhood. Suppose that* y *is a local minimizer to the isoperimetric problem given by* (3)*,* (4) *and* (14)*. Assume that there are functions* $\mathbf{h}^1, \ldots, \mathbf{h}^r \in \mathbf{D}$ *such that*

$$A = (a_{kl}), \quad a_{kl} := \delta\mathcal{G}^k(\boldsymbol{y}; \boldsymbol{h}^l), \text{ has maximal rank } r. \tag{15}$$

Then there exist constants $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ *such that the functional*

$$\mathcal{F} := \mathcal{J} - \sum_{j=1}^{r} \lambda_j \mathcal{G}^j$$

satisfies

$$\delta\mathcal{F}(\mathbf{y}; \mathbf{h}) = 0 \tag{16}$$

for all $\mathbf{h} \in \mathbf{D}$*.*

Theorem 18. *Let assumptions of Theorem 17 hold. If* y *is a local minimizer to Problem 2, then* y *satisfies the system of* N *fractional Euler–Lagrange equations*

$$\partial_i F_\lambda \{y\}_\gamma^{\alpha,\beta}(x) - \frac{d}{dx}\partial_{N+i} F_\lambda \{y\}_\gamma^{\alpha,\beta}(x) + D_{1-\gamma}^{\beta,\alpha}\partial_{2N+i} F_\lambda \{y\}_\gamma^{\alpha,\beta}(x) = 0, \tag{17}$$

$i = 2, \ldots, N+1$*, for all* $x \in [a,b]$*, where function* $F : [a,b] \times \mathbb{R}^{3N} \times \mathbb{R}^r \to \mathbb{R}$ *is defined by*

$$F_\lambda \{y\}_\gamma^{\alpha,\beta}(x) := L\,[y]_\gamma^{\alpha,\beta}(x) - \sum_{j=1}^{r} \lambda_j G^j\,[y]_\gamma^{\alpha,\beta}(x).$$

Proof. Under assumptions of Theorem 17, the equation (16) is fulfilled for every $\mathbf{h} \in \mathbf{D}$. Consider a function $\mathbf{h}$ such that $\mathbf{h}(a) = \mathbf{h}(b) = 0$. Then,

$$\begin{aligned} 0 = \delta\mathcal{F}(y; \mathbf{h}) &= \left.\frac{\partial}{\partial\epsilon}\mathcal{F}(y + \epsilon\mathbf{h})\right|_{\epsilon=0} \\ &= \int_a^b \left[\sum_{i=2}^{N+1} \partial_i\, F_\lambda \{y\}_\gamma^{\alpha,\beta}(x) h_{i-1}(x) + \sum_{i=2}^{N+1} \partial_{N+i} F_\lambda \{y\}_\gamma^{\alpha,\beta}(x) \frac{d}{dx} h_{i-1}(x) \right. \\ &\qquad \left. + \sum_{i=2}^{N+1} \partial_{2N+i} F_\lambda \{y\}_\gamma^{\alpha,\beta}(x)\, {}^C D_\gamma^{\alpha,\beta} h_{i-1}(x)\right] dx. \end{aligned}$$

Using the classical and the integration by parts formula (1), and applying the fundamental lemma of the calculus of variations in a similar way as in the proof of Theorem 13, we obtain (17). $\square$

Suppose now, that constraints (14) are characterized by inequalities

$$\mathcal{G}(y) = \int_a^b G^j [y]_\gamma^{\alpha,\beta}(x) dx \leq \xi_j, \quad j = 1, \ldots, r.$$

In this case we can set

$$\int_a^b \left(G^j [y]_\gamma^{\alpha,\beta}(x) - \frac{\xi_j}{b-a} \right) dx + \int_a^b (\phi_j(x))^2 dx = 0,$$

$j = 1, \ldots, r$, where ϕ_j have the same continuity properties as y_i. Therefore, we obtain the following problem: minimize the functional

$$\hat{\mathcal{J}}(y) = \int_a^b \hat{L}\left(x, \mathbf{y}(x), y'(x), {}^C D_\gamma^{\alpha,\beta} \mathbf{y}(x), \phi(x)\right) dx,$$

where $\phi(x) = [\phi_1(x), \ldots, \phi_r(x)]$, subject to r isoperimetric constraints

$$\int_a^b \left[G^j [y]_\gamma^{\alpha,\beta}(x) - \frac{\xi_j}{b-a} + (\phi_j(x))^2 \right] dx = 0, \quad j = 1, \ldots, r,$$

and boundary conditions (4). Assuming that assumptions of Theorem 18 are satisfied, we conclude that there exist constants $\lambda_j \in \mathbb{R}$, $j = 1, \ldots, r$, for which the system of N equations

$$\begin{aligned} &\partial_i \hat{F}\left(x, y(x), y'(x), {}^C D_\gamma^{\alpha,\beta} \mathbf{y}(x), \lambda_1, \ldots, \lambda_r, \phi(x)\right) \\ &\quad - \frac{d}{dx} \partial_{N+i} \hat{F}\left(x, y(x), y'(x), {}^C D_\gamma^{\alpha,\beta} y(x), \lambda_1, \ldots, \lambda_r, \phi(x)\right) \\ &\quad + D_{1-\gamma}^{\beta,\alpha} \partial_{2N+i} \hat{F}\left(x, y(x), y'(x), {}^C D_\gamma^{\alpha,\beta} \mathbf{y}(x), \lambda_1, \ldots, \lambda_r, \phi(x)\right) = 0, \end{aligned} \tag{18}$$

$i = 2, \ldots, N+1$, where $\hat{F} = \hat{L} + \sum_{j=1}^{r} \lambda_j \left(G^j - \frac{\xi_j}{b-a} + \phi_j^2 \right)$ and

$$\lambda_j \phi_j(x) = 0, \quad j = 1, \ldots, r, \tag{19}$$

hold for all $x \in [a, b]$. Note that it is enough to assume that the regularity condition (15) holds for the constraints which are active at the local minimizer $\mathbf{y}$. Indeed, suppose that $l < r$ constraints, say $\mathcal{G}^1, \ldots, \mathcal{G}^l$ for simplicity, are active at the local minimizer $\mathbf{y}$, and there are functions $\mathbf{h}^1, \ldots, \mathbf{h}^l \in \mathbf{D}$ such that the matrix $B = (b_{kj})$, $b_{k,j} := \delta \mathcal{G}^k \left(y; \mathbf{h}^j\right)$, $k, j = 1, \ldots, l < r$ has maximal rank l.

Since the inequality constraints $\mathcal{G}^{l+1}, \ldots, \mathcal{G}^r$ are inactive, the condition (19) is trivially satisfied by taking $\lambda_{l+1} = \ldots = \lambda_r = 0$. On the other hand, since the inequality constraints $\mathcal{G}^1, \ldots, \mathcal{G}^l$ are active and satisfy the regularity condition (15) at $\mathbf{y}$, the conclusion that there exist constants $\lambda_j \in \mathbb{R}$, $j = 1, \ldots, r$, such that (18) holds, follow from Theorem 18. Moreover, (19) is trivially satisfied for $j = 1, \ldots, l$.

4 An Illustrative Example

Let $\alpha \in (0,1)$, $N = 1$, $\gamma = 1$, and $\xi \in \mathbb{R}$. Consider the following fractional isoperimetric problem:

$$\begin{gathered}
\mathcal{J}(y) = \int_0^1 \left(y'(x) + {}_0^C D_x^\alpha y(x)\right)^2 dx \longrightarrow \min \\
\mathcal{G}(y) = \int_0^1 \left(y'(x) + {}_0^C D_x^\alpha y(x)\right) dx = \xi \qquad (20) \\
y(0) = 0\,,\ y(1) = \int_0^1 E_{1-\alpha}\left(-(1-t)^{1-\alpha}\right) \xi dt.
\end{gathered}$$

In this problem we make use of the Mittag-Leffler function $E_\alpha(z)$. We recall that the Mittag-Leffler function is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}\,.$$

This function appears naturally in the solution of fractional differential equations, as a generalization of the exponential function [9]. Indeed, while a linear second order ordinary differential equation with constant coefficients presents an exponential function as solution, in the fractional case the Mittag–Leffler functions emerge [17].

In our example (20) the function F of Theorem 18 is given by

$$F(x, y, y', {}_0^C D_x^\alpha y, \lambda) = \left(y' + {}_0^C D_x^\alpha y\right)^2 - \lambda\left(y' + {}_0^C D_x^\alpha y\right).$$

One can easily check that y such that

$$y(x) = \int_0^x E_{1-\alpha}\left(-(x-t)^{1-\alpha}\right) \xi dt \qquad (21)$$

- is not an extremal for $\mathcal{G}$;
- satisfies $y' + {}_0^C D_x^\alpha y = \xi$ (see, e.g., [17, p. 328, Example 5.24]).

Moreover, (21) satisfies the Euler–Lagrange equations (17) for $\lambda = 2\xi$, i.e.,

$$-\frac{d}{dx}\left(2\left(y' + {}_0^C D_x^\alpha y\right) - 2\xi\right) + {}_x D_1^\alpha \left(2\left(y' + {}_0^C D_x^\alpha y\right) - 2\xi\right) = 0.$$

We conclude that (21) is an extremal for the isoperimetric problem (20).

Remark 19. *When $\alpha \to 1$ the isoperimetric constraint is redundant with the boundary conditions, and the fractional isoperimetric problem* (20) *simplifies to the classical variational problem*

$$\begin{gathered}\mathcal{J}(y) = 4\int_0^1 (y'(x))^2 dx \longrightarrow \min \\ y(0) = 0\,, \quad y(1) = \frac{\xi}{2}.\end{gathered} \tag{22}$$

Our fractional extremal (21) *gives $y(x) = \frac{\xi}{2}x$ for $i = 1, \ldots, N$, which is exactly the minimizer of* (22).

Remark 20. *Choose $\xi = 1$. When $\alpha \to 0$ one gets from* (20) *the classical isoperimetric problem*

$$\begin{gathered}\mathcal{J}(y) = \int_0^1 \left(y'(x) + y(x)\right)^2 dx \longrightarrow \min \\ \mathcal{G}(y) = \int_0^1 y(x)dx = \frac{1}{e} \\ y(0) = 0 \quad y_i(1) = 1 - \frac{1}{\mathrm{e}}.\end{gathered} \tag{23}$$

Our extremal (21) *is then reduced to the classical extremal $y(x) = 1 - \mathrm{e}^{-x}$ of the isoperimetric problem* (23).

Remark 21. *Let $\alpha = \frac{1}{2}$. Then* (20) *gives the following fractional isoperimetric problem:*

$$\begin{gathered}\mathcal{J}(y) = \int_0^1 \left(y'(x) + {}_0^C D_x^{\frac{1}{2}} y(x)\right)^2 dx \longrightarrow \min \\ \mathcal{G}(y) = \int_0^1 \left(y'(x) + {}_0^C D_x^{\frac{1}{2}} y(x)\right) dx = \xi \\ y(0) = 0\,, \quad y(1) = \xi \left(\mathrm{erfc}(1)\mathrm{e} + \frac{2}{\sqrt{\pi}} - 1\right),\end{gathered} \tag{24}$$

where erfc *is the complementary error function defined by*

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty exp(-t^2)dt.$$

The extremal (21) *for the particular fractional isoperimetric problem* (24) *is*

$$y(x) = \xi \left(\mathrm{e}^x \mathrm{erfc}(\sqrt{x}) + \frac{2\sqrt{x}}{\sqrt{\pi}} - 1\right).$$

Acknowledgements

This work was partially supported by the *Portuguese Foundation for Science and Technology* (FCT) through the *Center for Research and Development in*

Mathematics and Applications (CIDMA). TO is also supported by FCT through the Ph.D. fellowship SFRH/BD/33865/2009; ABM by BUT grant S/WI/1/08; and DFMT through the project UTAustin/MAT/0057/2008.

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